Exercise 59

Show that the function $f(x) = |x - 6|$ is not differentiable at 6. Find a formula for f' and sketch its graph.

Solution

Use the definition of the derivative to find f'.

$$\begin{aligned}
f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0} \frac{|(x+h) - 6| - |x-6|}{h} \\
&= \lim_{h\to 0} \frac{|x+h-6| - |x-6|}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{(x+h-6)^2} - \sqrt{(x-6)^2}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{(x+h-6)^2} - \sqrt{(x-6)^2}}{h} \cdot \frac{\sqrt{(x+h-6)^2} + \sqrt{(x-6)^2}}{\sqrt{(x+h-6)^2} + \sqrt{(x-6)^2}} \\
&= \lim_{h\to 0} \frac{(x+h-6)^2 - (x-6)^2}{h\left[\sqrt{(x+h-6)^2} + \sqrt{(x-6)^2}\right]} \\
&= \lim_{h\to 0} \frac{[(x+h)^2 - 12(x+h) + 36] - (x^2 - 12x + 36)}{h\left[\sqrt{(x+h-6)^2} + \sqrt{(x-6)^2}\right]} \\
&= \lim_{h\to 0} \frac{[(x^2 + 2xh + h^2) - 12x - 12h + 36] - x^2 + 12x - 36}{h\left[\sqrt{(x+h-6)^2} + \sqrt{(x-6)^2}\right]} \\
&= \lim_{h\to 0} \frac{2xh + h^2 - 12h}{h\left[\sqrt{(x+h-6)^2} + \sqrt{(x-6)^2}\right]} \\
&= \lim_{h\to 0} \frac{2x + h - 12}{\sqrt{(x+h-6)^2} + \sqrt{(x-6)^2}} \\
&= \frac{2x - 12}{\sqrt{(x-6)^2} + \sqrt{(x-6)^2}} \\
&= \frac{2(x-6)}{2\sqrt{(x-6)^2}} \\
&= \frac{x-6}{\sqrt{(x-6)^2}} \\
&= \frac{x-6}{|x-6|} \\
&= \operatorname{sgn}(x-6)
\end{aligned}$$

$f(x) = |x-6|$ is not differentiable at 6 because there's $|x-6|$ in the denominator, and for any rational function the denominator cannot be zero.

$$|x-6| \neq 0$$

$$x-6 \neq 0$$

$$x \neq 6$$

The domain of $f'(x)$ is $\{x \mid x \neq 6\}$. A graph of $f(x)$ and $f'(x)$ versus x is shown below.